State-feedback $\mathcal{H}_\infty$ control $\Rightarrow \frac{\|z\|_2}{\|w\|_2} < \gamma$

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Consider a closed-loop system composed of a plant

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right], \tag{1}$$

and a state-feedback control $u = -Fx$. Suppose the followings for simplicity:

- (A, B_2) : stabilizable
- (C_1, A) : detectable
- D_{12} : column full rank
- $G_{12}(s)$ has no zero on the imaginary axis
- $D_{12}^T C_1 = 0$, $D_{12}^T D_{12} = R > 0$, $C_1^T C_1 = Q > 0$

Then, the following statement holds.

Lemma 1 *There exist a state-feedback gain F such that*

(i) the closed-loop is stable

(ii) the closed-loop $\mathcal{H}_\infty$ norm is less than γ

if and only if there exists a real number $\epsilon > 0$ such that the following ARE has a solution $P > 0$:

$$A^T P + PA + P\left(\frac{1}{\gamma^2} B_1 B_1^T - B_2 R^{-1} B_2^T\right) P + C_1^T C_1 + \epsilon I = 0. \tag{2}$$

Moreover, the state-feedback gain F is given by

$$F = R^{-1} B_2^T P. \tag{3}$$

Proof 1 *(⇓) : omit*
(⇑) (i): omit (similarly proved to LQR)
(⇑) (ii):

$$\int_0^\infty \{z^T z - \gamma^2 w^T w\}\, dt = \int_0^\infty \{x^T Q x + u^T R u - \gamma^2 w^T w\}\, dt \tag{4}$$

$$= \int_0^\infty \Bigg[x^T \left\{ -A^T P - PA + PB_2 R^{-1} B_2^T P - \frac{1}{\gamma^2} PB_1 B_1^T P - \epsilon I \right\} x + u^T R u - \gamma^2 w^T w - u^T B_2^T P x - x^T P B_2 u$$
$$+ u^T B_2^T P x + x^T P B_2 u - w^T B_1^T P x - x^T P B_1 w + w^T B_1^T P x + x^T P B_1 w \Bigg] dt \tag{5}$$

... the last 8 terms are canceled out in summation

$$= \int_0^\infty \Bigg[-\{x^T A^T + w^T B_1^T + u^T B_2^T\} P x - x^T P \{Ax + B_1 w + B_2 u\} + \{x^T P B_2 R^{-1} + u^T\} R \{R^{-1} B_2^T P x + u\}$$
$$- \left\{\gamma w^T - \frac{1}{\gamma} x^T P B_1\right\}\left\{\gamma w - \frac{1}{\gamma} B_1^T P x\right\} - \epsilon x^T x \Bigg] dt \tag{6}$$

$$
\begin{aligned}
& \quad \textit{... each colored terms are collected} \\
&= -\int_0^\infty \underline{\dot{x}^T P x} dt - \int_0^\infty \underline{\underline{x^T P \dot{x}}} dt - \int_0^\infty \left\{\gamma w^T - \frac{1}{\gamma} x^T P B_1\right\}\left\{\gamma w - \frac{1}{\gamma} B_1^T P x\right\} dt - \epsilon \int_0^\infty x^T x dt \qquad (7) \\
& \quad \textit{... from } \dot{x} = Ax + B_1 w + B_2 u \textit{ and } u = -R^{-1} B_2^T P x \\
&= -\left[x^T P x\right]_0^\infty + \int_0^\infty x^T P \dot{x} dt - \int_0^\infty \underline{\underline{x^T P \dot{x}}} dt - \int_0^\infty \left\{\gamma w^T - \frac{1}{\gamma} x^T P B_1\right\}\left\{\gamma w - \frac{1}{\gamma} B_1^T P x\right\} - \epsilon \int_0^\infty x^T x dt \qquad (8) \\
& \quad \textit{... the 1st term is integrated in parts.} \\
&= x(0)^T P x(0) - \int_0^\infty \left\{\gamma w^T - \frac{1}{\gamma} x^T P B_1\right\}\left\{\gamma w - \frac{1}{\gamma} B_1^T P x\right\} - \epsilon \int_0^\infty x^T x dt \qquad (9) \\
& \quad \textit{... the 2nd and 3rd terms are canceled. the 1st term is from } x(\infty) = 0. \\
&< x(0)^T P x(0) = 0. \qquad (10)
\end{aligned}
$$

Note that the above holds for any $w(t)$. *Thus,*

$$
\sup_{w(t)} \frac{\|z(t)\|_2}{\|w(t)\|_2} < \gamma \qquad (11)
$$

holds, which completes the proof.

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