LQR $\Rightarrow J$ is minimized

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Lemma 1 *For a given plant*

$$\dot{x} = Ax + Bu, \tag{1}$$

let $P > 0$ be the solution of the algebraic Riccati equation

$$PA + A^T P + Q - PBR^{-1}B^T P = 0 \tag{2}$$

where $Q \geq 0$ and $R > 0$ are weighting matrices for the cost function J defined as

$$J = \int_0^\infty \{x^T Qx + u^T Ru\}\, dt. \tag{3}$$

Suppose that the control input u is given as follows:

$$u(t) = -Fx(t), \quad F := R^{-1}B^T P. \tag{4}$$

Then, the following conditions hold.

(i) The closed-loop system is stable.

(ii) J is minimized to have $J_{\min} = x^T(0)Px(0)$, where $x(0)$ is the initial state.

Proof 1 *(i) : The following equation holds from the assumptions*

$$P(A - BF) + (A - BF)^T P = -Q - F^T RF, \tag{5}$$

where $F = R^{-1}B^T P$. Since the right hand side matrix is negative definite, the following equation holds:

$$A_{cl}^T P + PA_{cl} < 0 \tag{6}$$

where $A_{cl} := A - BF$. Thus, $\dot{x} = A_{cl}x$ is stable according to Lemma 2 (ii) $\Rightarrow$ (i).
(ii) :

$$\begin{aligned}
J &= \int_0^\infty \{x^T Qx + u^T Ru\}\, dt && (7)\\
&= \int_0^\infty \left[x^T \left\{-A^T P - PA + PBR^{-1}B^T P\right\} x + u^T Ru - u^T B^T Px - x^T PBu + u^T B^T Px + x^T PBu\right] dt && (8)\\
&\quad \text{... the last 4 terms are 0 in summation.}\\
&= \int_0^\infty \left[-\{x^T A^T + u^T B^T\} Px - x^T P\{Ax + Bu\} + \{x^T PBR^{-1} + u^T\} R \{R^{-1}B^T Px + u\}\right] dt && (9)\\
&\quad \text{... each colored terms are collected}\\
&= -\int_0^\infty \dot{x}^T Px dt - \int_0^\infty x^T P\dot{x} dt + \int_0^\infty \{x^T PBR^{-1} + u^T\} R \{R^{-1}B^T Px + u\} dt && (10)\\
&\quad \text{... from } \dot{x} = Ax + Bu\\
&= -\left[x^T Px\right]_0^\infty + \int_0^\infty x^T P\dot{x} dt - \int_0^\infty x^T P\dot{x} dt + \int_0^\infty \{x^T PBR^{-1} + u^T\} R \{R^{-1}B^T Px + u\} dt && (11)\\
&\quad \text{... the 1st term is integrated in parts.}\\
&= x(0)^T Px(0) + \int_0^\infty \{x^T PBR^{-1} + u^T\} R \{R^{-1}B^T Px + u\} dt && (12)\\
&\quad \text{... the 2nd and 3rd terms are canceled. the 1st term is from } x(\infty) = 0.
\end{aligned}$$

Note that the 1st term in the last equation does not depend on u. Thus, it is necessary to set $u = -R^{-1}B^T Px$ so that J has the minimized value of $x(0)^T Px(0)$, which completes the proof. □

Lemma 2 *Suppose A is given real square matrix. The following conditions are equivalent.*

(i) A is stable, i.e. the following equation holds:

$$\lambda_i(A) + \bar{\lambda}_i(A) < 0 \quad {}^{\forall} i \tag{13}$$

where $\lambda_i(A)$ is the eigenvalue of A.

(ii) There exist a positive definite matrix P such that

$$PA + A^T P < 0. \tag{14}$$

Proof 2 *(ii)⇒(i) : Let an eigenvalue of A be λ, and the corresponding eigenvector be v. By definition, $v \neq 0$ and $Av = \lambda v$ are satisfied. Thus, the following hold:*

$$v^*(PA + A^T P)v = \lambda v^* Pv + \bar{\lambda} v^* Pv = (\lambda + \bar{\lambda}) v^* Pv < 0. \tag{15}$$

According to the positive definiteness of P, $v^* Pv > 0$ is satisfied for any non-zero complex vector v. Therefore $\lambda + \bar{\lambda} < 0$ holds, which completes the proof of (i).
(i)⇒(ii) : omitted